Universal spectral parameter-dependent Lax operators for the Drinfeld double of the dihedral group D_3

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Abstract

Two universal spectral parameter-dependent Lax operators are presented in terms of the elements of the Drinfeld double $D(D_3)$ of the dihedral group D_3. Applying representations of $D(D_3)$ to these yields matrix solutions of the Yang–Baxter equation with a spectral parameter.

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1. Introduction

The Yang–Baxter equation arises in several areas of mathematical physics. In the framework of quasi-triangular Hopf algebras, Drinfeld's double construction [1] provides a systematic means of constructing solutions for the Yang–Baxter equation. The double construction produces a canonical element, known as the universal R-matrix, which solves the Yang–Baxter equation algebraically. For applications to areas such as statistical mechanics lattice models [2, 3] and integrable quantum systems [4, 5], solutions which depend on a spectral parameter are necessary (which we hereafter refer to as *parametric* solutions in contrast to *constant* solutions). For some years quantum algebras (deformations of the universal enveloping algebras of Lie algebras) were a fertile field of study as an application of the Drinfeld construction, whereby many parametric solutions of the Yang–Baxter equation were obtained. Specifically, through use of an evaluation homomorphism from an affine quantum algebra to a non-affine subalgebra it is in principle possible to construct algebraic parametric solutions of the Yang–Baxter equation. Such a solution is a powerful tool in constructing a general class of matrix solutions by applying different algebra representations to the algebraic solution. However this procedure is technically challenging and only a few cases for low-rank algebras have been made fully explicit [6, 7]. The more common practice was to start with a constant matrix solution of the Yang–Baxter equation and then determine an analogous

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parametric form, a process which is commonly known as *Baxterization* [8–15]. An alternative approach was to study the parametric Lax operator which is defined in the tensor product of an algebra and one of its representations. However the only instances of quantum algebras for which a solution for the parametric Lax operator is known are $U_q(gl(n))$ [16] and its $\mathbb{Z}_2$-graded generalization $U_q(gl(m|n))$ [17].

While quantum algebras, being a seminal class of quasi-triangular Hopf algebras, generated substantial activity as a tool for solving the parametric Yang–Baxter equation there has not been as much interest in the study of other quasi-triangular Hopf algebras for this purpose. One such example is the class of Drinfeld doubles of finite group algebras. These algebras have been studied as appropriate symmetry algebras for the description of non-Abelian anions where the conjugacy classes and centralizer subgroups of the finite-group label generalized notions of the magnetic and electric charges, respectively [18]. The study of non-Abelian anions is currently active in relation to proposals for topological quantum computation [19–21]. A recent analysis of an integrable one-dimensional quantum system for non-Abelian Fibonacci anions with three-body interactions can be found in [22]. We have previously shown that by using the Drinfeld double of the simplest non-Abelian finite group D_3, yielding the quasi-triangular Hopf algebra denoted $D(D_3)$, there is a solution of the Yang–Baxter equation which naturally leads to an integrable model of non-Abelian anions with two-body interactions [23]. Moreover this solution can be extended to the Drinfeld doubles of general dihedral groups $D(D_n)$ [24]. Here we continue work in this direction by presenting two explicit universal parametric Lax operators for $D(D_3)$, one associated with a two-dimensional representation and the other for a three-dimensional representation. This result suggests the possibility of a universal parametric R-matrix $R(x) \in D(D_3) \otimes D(D_3)$, and motivates future study. We expect that a comprehensive understanding of the solutions of the Yang–Baxter equation with symmetries given by the Drinfeld doubles of finite groups will ultimately provide important insights into interacting systems with non-Abelian anionic degrees of freedom.

2. The Drinfeld double of the dihedral group D_3

The dihedral group D_3 (also known as the symmetric group S_3) represents the symmetries of a triangle, and has two generators σ, τ satisfying

$$\sigma^3 = e, \qquad \tau^2 = e, \qquad \tau\sigma = \sigma^2\tau,$$

where e denotes the identity. The Drinfeld double [1] of D_3, denoted $D(D_3)$, has basis

$$\{gh^* | g, h \in D_3\},$$

where g are the group elements and g^* are dual elements. This gives an algebra of dimension 36. Multiplication of dual elements is given by

$$g^*h^* = \delta(g, h)g^*, \tag{1}$$

where δ is the usual delta function. The products h^*g are computed using

$$h^*g = g(g^{-1}hg)^* \qquad gh^* = (ghg^{-1})^*g. \tag{2}$$

The algebra $D(D_3)$ becomes a Hopf algebra by imposing the following coproduct, antipode and co-unit, respectively:

$$\Delta(gh^*) = \sum_{k \in G} g(k^{-1}h)^* \otimes gk^* = \sum_{k \in G} gk^* \otimes g(hk^{-1})^*,$$

$$S(gh^*) = (h^{-1})^*g^{-1} = g^{-1}(gh^{-1}g^{-1})^*,$$

$$\varepsilon(gh^*) = \delta(h, e), \qquad \forall g, h \in D_3.$$

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Note that we identify $g\varepsilon$ with g and eg^* with g^* for all $g \in G$. The universal R-matrix is given by

$$\mathcal{R} = \sum_{g \in D_3} g \otimes g^*.$$

This can easily be shown to satisfy the defining relations for a quasi-triangular Hopf algebra:

$$\mathcal{R}\Delta(a) = \Delta^T(a)\mathcal{R}, \qquad \forall a \in D(G), \tag{3}$$

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \tag{4}$$

$$(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{5}$$

where Δ^T is the opposite coproduct

$$\Delta^T(gh^*) = \sum_{k \in G} gk^* \otimes g(k^{-1}h)^* = \sum_{k \in G} g(kh^{-1})^* \otimes gk^*.$$

It follows from the relations (3), (4), (5) that $\mathcal{R}$ is a solution of the constant Yang–Baxter equation, which is given by

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

The Yang–Baxter equation operates on the three-fold tensor product space $D(D_3)^{\otimes 3}$, and $\mathcal{R}_{ij}$ indicates that $\mathcal{R}$ is acting on the ith and jth spaces. For example, $\mathcal{R}_{12} = \mathcal{R} \otimes I$, where $I \in D(D_3)$ denotes the identity operator.

Throughout this paper we also use the following variant of the Yang–Baxter equation involving a spectral parameter:

$$R_{12}(x/y)R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(x/y).$$

Here the Yang–Baxter equation acts on End $(V \otimes V \otimes V)$ where V is a vector space and $R(x) \in$ End $(V \otimes V)$. We refer to a solution to this equation as an R-matrix, and in particular we use the solutions given in [23].

Given an R-matrix $R(x) \in$ End $(V \otimes V)$, we define $\mathfrak{L}(x) \in$ End $V \otimes D(D_3)$ to be a *universal Lax operator* if $\mathfrak{L}(x)$ satisfies the following equation:

$$R_{12}(x/y)\mathfrak{L}_{13}(x)\mathfrak{L}_{23}(y) = \mathfrak{L}_{23}(y)\mathfrak{L}_{13}(x)R_{12}(x/y). \tag{6}$$

By applying a representation to the third space we obtain the well-known $RLL = LLR$ equation with spectral parameter, which acts on End $V \otimes V \otimes W$ and is given by

$$R_{12}(x/y)L_{13}(x)L_{23}(y) = L_{23}(y)L_{13}(x)R_{12}(x/y). \tag{7}$$

Here $R(x) \in$ End $(V \otimes V)$ is an R-matrix and $L(x) \in$ End $(V \otimes W)$ is known as a (matrix) Lax operator.

2.1. Representation theory of $D(D_3)$

Throughout I_n will denote the n-dimensional identity matrix, while E^i_j denotes the basis matrix with 1 in the (i, j) position and zeros elsewhere. The Hopf algebra $D(D_3)$ has two one-dimensional irreducible representations (irreps), four two-dimensional irreps and two three-dimensional irreps. They are given by

- one-dimensional irreps

$$\pi_{(1,\pm)}\sigma = 1, \qquad \pi_{(1,\pm)}(\tau) = \pm 1, \qquad \pi_{(1,\pm)}(g^*) = \delta(g, e).$$

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- two-dimensional irreps
  Set ω to be the cube root of unity $\omega = e^{2\pi i/3}$. Then

$$\pi_{(2,e)}(\sigma) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \qquad \pi_{(2,e)}(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \pi_{(2,e)}(g^*) = \delta(g,e)I_2$$

  and

$$\pi_{(2,i)}(\sigma) = \begin{pmatrix} \omega^i & 0 \\ 0 & \omega^{-i} \end{pmatrix}, \qquad \pi_{(2,i)}(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\pi_{(2,i)}(g^*) = \delta(g,\sigma)E_1^1 + \delta(g,\sigma^{-1})E_2^2$$

  for $i = 0, 1, 2$.
- three-dimensional irreps

$$\pi_{(3,\pm)}(\sigma) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \pi_{(3,\pm)}(\tau) = \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\pi_{(3,\pm)}((\sigma^i)^*) = 0, \qquad \pi_{(3,\pm)}((\sigma^i\tau)^*) = E_{i+1}^{i+1}, \qquad i = 0, 1, 2.$$

Note that in each irrep, the non-zero dual elements correspond to precisely one conjugacy class of D_3. Throughout this paper we denote the module associated with an irrep π_Λ by V_Λ.

We utilize the R-matrices obtained in [23]. Explicitly we have the R-matrix $R_{(2,1)}(x) \in$ End $(V_{(2,1)} \otimes V_{(2,1)})$ given by

$$R_{(2,1)}(x) = \begin{pmatrix} \omega - x^2 & 0 & 0 & 0 \\ 0 & -\omega^{-1}(x^2-1) & (\omega-1)x & 0 \\ 0 & (\omega-1)x & -\omega^{-1}(x^2-1) & 0 \\ 0 & 0 & 0 & \omega - x^2 \end{pmatrix},$$

which is simply the six-vertex model at a cube root of unity. Further, we have the R-matrix $R_{(3,+)}(x) \in$ End $(V_{(3,+)} \otimes V_{(3,+)})$ given by

$$R_{(3,+)}(x) =$$
$$\begin{pmatrix} x^2-x+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x(x-1) & x & 0 & 0 & 0 & 1-x & 0 \\ 0 & x(x-1) & 0 & 0 & 0 & 1-x & x & 0 & 0 \\ 0 & x & 0 & 0 & 0 & x(x-1) & 1-x & 0 & 0 \\ 0 & 0 & 0 & 0 & x^2-x+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-x & x(x-1) & 0 & 0 & 0 & x & 0 \\ 0 & 0 & x & 1-x & 0 & 0 & 0 & x(x-1) & 0 \\ 0 & 1-x & 0 & 0 & 0 & x & x(x-1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2-x+1 \end{pmatrix}.$$

3. Two universal Lax operators for $D(D_3)$

The Lax operators are constructed by assuming an ansatz involving unknown functions and central elements, substituting into equation (6), and then solving to determine suitable forms for the functions and central elements in the ansatz. First we remark that for any finite group G it can be shown that

$$\mathcal{P} = \sum_{g,h \in G} gh^* \otimes h^*g^{-1}$$

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is a universal permutation operator satisfying

$$\mathcal{P}(x \otimes y) = (y \otimes x)\mathcal{P} \tag{8}$$

for all $x, y \in D(G)$. Relation (8) can be proved on the basis elements $x = gh^*, y = jk^*$ with $g, h, j, k \in G$ simply by using equations (1) and (2). The result holds for arbitrary $x, y \in D(G)$ by linearity. Next, given a representation π of $D(D_3)$ the ansatz adopted for the Lax operator is

$$\mathcal{L}(x) = (\pi \otimes \text{id})\big(\mathcal{R} + f(x)(I \otimes c)\mathcal{P} + g(x)(I \otimes d)\mathcal{R}_{21}^{-1}\big)$$

where c and d are central elements of $D(D_3)$. Below we present the results for the representations $\pi_{(2,1)}$ and $\pi_{(3,+)}$.

3.1. Universal Lax operator associated with a two-dimensional representation

A Lax operator $\mathcal{L}(x) \in \text{End}(V_{(2,1)}) \otimes D(D_3)$ is given by

$$\mathcal{L}(x)$$
$$= \sum_{j \in \mathbb{Z}_3} \begin{pmatrix} \omega x \sigma^{-1}(\sigma^j \tau)^* + (\omega^j - \omega x^2 c_1^2 \sigma^{-1})(\sigma^j)^* & \omega^j(\sigma^j \tau)^* + \frac{1}{3}(\omega - 1)c_1 x \omega^j \sigma^j \tau \sigma^* \\ \omega^j(\sigma^{-j}\tau)^* + \frac{1}{3}(\omega - 1)c_1 x \omega^j \sigma^{-j}\tau(\sigma^{-1})^* & \omega x \sigma(\sigma^j \tau)^* + (\omega^j - \omega x^2 c_1^2 \sigma)(\sigma^{-j})^* \end{pmatrix}.$$

Here c_1 is the following central element of $D(D_3)$:

$$c_1 = \tfrac{1}{3}[(2e - \sigma - \sigma^{-1})(\sigma^* + (\sigma^{-1})^*].$$

By direct calculation we have verified that $\mathcal{L}(x)$ satisfies the algebraic relation (6) on $\text{End}(V_{(2,1)} \otimes V_{(2,1)}) \otimes D(D_3)$ with the R-matrix $R_{(2,1)}(x)$, and is hence a universal Lax operator. In the limit $x \to 0$, this Lax operator reduces to $(\pi_{(2,1)} \otimes \text{id})\mathcal{R}$ where $\mathcal{R}$ is the constant universal R-matrix given earlier.

Now defining $L_{(a,b)}(x) = (\text{id} \otimes \pi_{(a,b)})\mathcal{L}(x)$, we note that

$$L_{(2,e)}(x) = I_2 \otimes I_2,$$
$$L_{(2,0)}(x) = \text{diag}(\omega, \omega^{-1}, \omega^{-1}, \omega),$$
$$L_{(2,1)}(x) = R_{(2,1)}(x),$$
$$L_{(2,2)}(x) = \text{diag}(\omega - \omega^{-1}x^2, \omega^{-1} - x^2, \omega^{-1} - x^2, \omega - \omega^{-1}x^2),$$

$$L_{(3,\pm)}(x) = \begin{pmatrix} 0 & 0 & \omega x & 1 & 0 & 0 \\ \omega x & 0 & 0 & 0 & \omega & 0 \\ 0 & \omega x & 0 & 0 & 0 & \omega^{-1} \\ 1 & 0 & 0 & 0 & \omega x & 0 \\ 0 & \omega^{-1} & 0 & 0 & 0 & \omega x \\ 0 & 0 & \omega & \omega x & 0 & 0 \end{pmatrix}$$

It is straightforward to verify that these matrices satisfy the $RLL = LLR$ relation (7) on $V_{(2,1)} \otimes V_{(2,1)} \otimes V_\Lambda$ for all irreducible modules V_Λ, which again confirms that $\mathcal{L}(x)$ is a universal Lax operator.

3.2. Universal Lax operator associated with a three-dimensional representation

A Lax operator $\mathcal{L}(x) \in \text{End}(V_{(3,+)}) \otimes D(D_3)$ is given by

$$\mathcal{L}(x) = \sum_{i,j=1,2,3} E_j^i \otimes \big[(1-x)[((\sigma^{j-i})^* + (\sigma^{2-(i+j)}\tau)^*]$$
$$+ x(x-1)c_2 \delta_j^i \sigma^{i-1}\tau + x\sigma^{i-j}(\sigma^{i-1}\tau)^*\big].$$

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In the matrix form, this reads

$$\mathfrak{L}(x) = \begin{pmatrix} (1-x)[e^* + \tau^*] + c_2 x(x-1)\tau + x\tau^* & (1-x)[\sigma^* + (\sigma^{-1}\tau)^*] + x\sigma^{-1}\tau^* \\ (1-x)[(\sigma^{-1})^* + (\sigma^{-1}\tau)^*] + x\sigma(\sigma\tau)^* & (1-x)[e^* + (\sigma\tau)^*] + c_2 x(x-1)\sigma\tau + x(\sigma\tau)^* \\ (1-x)[\sigma^* + (\sigma\tau)^*] + x\sigma^{-1}(\sigma^{-1}\tau)^* & (1-x)[(\sigma^{-1})^* + \tau^*] + x\sigma(\sigma^{-1}\tau)^* \end{pmatrix}$$

$$\begin{pmatrix} (1-x)[(\sigma^{-1})^* + (\sigma\tau)^*] + x\sigma\tau^* \\ (1-x)[\sigma^* + \tau^*] + x\sigma^{-1}(\sigma\tau)^* \\ (1-x)[e^* + (\sigma^{-1}\tau)^*] + c_2 x(x-1)\sigma^{-1}\tau + x(\sigma^{-1}\tau)^* \end{pmatrix}.$$

Here c_2 is the following central element of $D(D_3)$:

$$c_2 = \frac{1}{3}[2e - \sigma - \sigma^{-1}](\sigma^* + (\sigma^{-1})^*) + \sum_{k \in \mathbb{Z}_3} \sigma^k \tau (\sigma^k \tau)^*.$$

We have also verified that the above satisfies the defining relation (6) for a universal Lax operator. Moreover, the expression again reduces to the constant solution of the Yang–Baxter equation in the limit $x \to 0$. That is, $\lim_{x \to 0} \mathfrak{L}(x) = (\pi_{(3,+)} \otimes \mathrm{id})\mathcal{R}$.

Setting $L_{(a,b)}(x) = (\mathrm{id} \otimes \pi_{(a,b)})\mathfrak{L}(x)$, we obtain

$$L_{(3,\pm)}(x) = R_{(3,+)}(x)$$

$$L_{(2,e)}(x) = (1-x)I_3 \otimes I_2,$$

$$L_{(2,j)}(x) = (1-x)$$

$$\times \begin{pmatrix} 0 & (1-\delta_0^j)x & 1 & 0 & 0 & 0 \\ (1-\delta_0^j)x & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & (1-\delta_0^j)x\omega^j & 1 & 0 \\ 0 & 1 & (1-\delta_0^j)x\omega^{-j} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & (1-\delta_0^j)x\omega^{-j} \\ 0 & 0 & 0 & 1 & (1-\delta_0^j)x\omega^j & 0 \end{pmatrix}.$$

It can be verified that the above matrices satisfy the $RLL = LLR$ relation (7) on $V_{(3,+)} \otimes V_{(3,+)} \otimes V_\Lambda$ for all irreducible modules Λ.

Acknowledgments

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